

## Urban shapes as fractals

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*Summary* The fractal dimension of a curve provides a measure of its irregularity and if this dimension is stable over many scales, a single process is assumed to be generating the curve. Here we measure the fractal dimensions of curves defining the shape of the urban area of Cardiff during a period of its most rapid growth from 1886 to 1922. The results of applying the traditional scaling model suggest a new model in which fractal dimension varies with scale. Applications of this model show a decline in dimension over all scales and through time.

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Boundaries partitioning complex systems from one another and from their environments reflect properties and processes which can be inferred from their morphology (Richter and Peitgen 1985). For example, transport and building technologies, social controls over development as well as physical constraints determine the irregularity of the boundary dividing urban from rural areas while the shape of a coastline reflects the action of a variety of geophysical processes. Conventional descriptions of such boundaries reveal that their length is a function of the scale at which they are measured (Richardson 1981; Steinhaus 1954). Mandelbrot (1967) has labelled such boundaries as fractal and has shown their dimension  $D$  to lie between 1 and 2, thus providing an index of their ruggedness or irregularity.

If this fractal dimension is stable over many scales, the morphology is consistent with a single set of processes operating at every scale and the phenomena is said to be self-similar. Measurements of the fractal dimensions of boundaries particularly coastlines (Kent and Wong 1982; Mandelbrot 1967; Nakano 1983; Richardson 1961) and fine particles (Flook 1978; Kaye, Leblanc and Abbot 1985; Orford and Whalley 1983) have revealed such phenomena to be multi-fractal with lower fractal dimensions at smaller scales. This is explicable in terms of different processes operating at different scales especially where man-made and natural processes combine (Kaye 1984). The importance of the fractal dimension thus consists in identifying the range of scales over which processes operate and changes in the scale at which such processes operate through time. It also enables changes in the morphological effects of self-similarity to be explored.

Here we will examine how the irregularity of the boundary of an urban area changes at different scales and through time, with the intention of inferring changes in the processes which condition urban growth in time and space. A multitude of processes determine the morphology of the city: the technology of building, patterns of land tenure, the size of building plots, the demand for residential space, the mobility of the population and the efficiency and availability of transport technology, all combine to shape the growth of a city. These processes manifest themselves at different scales, for example building technologies at smaller, transport at larger scales. It is a reasonable assumption that these processes are reflected in the boundary of the city, hence in its degree of irregularity and fractal dimension (Perkal 1966).

### Defining and measuring urban shapes

Three hypotheses concerning changes in the fractal dimension of urban boundaries are postulated. First, the boundary is multi-fractal across a range of scales. Second, as there is greater control over physical development at smaller scales, the fractal dimension is likely to decrease with scale. Third, the fractal dimension at smaller scales should decrease over time as greater controls over building technology and land development have been instituted. At larger scales, it is less clear how the fractal dimension changes although increasing mobility and accessibility imply it too will decrease through time.

These hypotheses have been tested by determining the fractal dimensions of the urban boundary of Cardiff in 1886, 1901 and 1922. These times were chosen because of the rapid urban growth of the city from a population of 80,000 to 230,000 during this period. This period also marked the development of the tramway system which began in 1872 and was complete by 1914 and it was the period when the predominant style of late Victorian worker housing gave way to more spacious suburban housing. The landed estates which dominated the form of development in Cardiff in the mid-19th century were no longer significant and the period represented the pinnacle of industrial prosperity in Cardiff which was ended by World War I (Daunton 1977).

The urban boundaries defined from 1 : 10560 scale Ordnance Survey maps in 1886, 1901 and 1922, were digitised to 1 mm accuracy, and are displayed in Figure 1. Considerable control was exercised in digitising to ensure the same level of detail was picked up from each map, thus minimising the possibility that the fractal dimension becomes an artifact of the observed data. Computing these dimensions involves two stages: first the length of the perimeter of each boundary is calculated by simulating a traverse of the curve at different scales and second, these perimeter estimates are related to their associated scales using a curve fitting procedure which yields the fractal dimension. The perimeter,  $P(\Delta x)$ , is measured using a simulation of Richardson's method of walking a pair of dividers around the curve, the step length of the dividers,  $\Delta x$ , being a measure of scale (Richardson 1961; Shelberg, Moellering and Lam 1982). This structured walk involves a protocol in which the simulation is started at any point on the curve, proceeding in both directions to the curve's end points, the last steps being a fraction of the fixed step size (Kaye 1978). The procedure is started at every digitised point on the curve and the perimeter taken as an average of each walk to remove any dependence on starting values. The method is extremely time-consuming, each pass of the method taking 65 minutes of CPU time on a Microvax II for a curve involving 4755 digitised points (the 1922 boundary).

The scales used in each walk varied from a step length  $\Delta x_0$  computed as the average of the chords linking the digitised points, to a scale which gave not less than eight chords, below which any approximation to the boundary was unacceptable. Thirty changes in scale were used and each scale was related to the lowest scale  $\Delta x_0$  by  $\Delta x_k = \Psi^k \Delta x_0, k = 0, 1, \dots, 30$ , where  $\Psi$  is a parameter controlling the size of the geometric change between scales. These scales ensure equal weighting of values in the log-log regressions based on the log-log plots of perimeter against scale. These are shown in Figure 2 for each boundary.

### Fractal models of urban boundaries

The fractal model relating perimeter to scale is based on

$$P(\Delta x) = a \Delta x^{-\beta} \quad (1)$$

where  $a$  and  $\beta$  are scaling constants. Mandelbrot (1967, 1983) has shown that  $\beta$  is related to the fractal dimension as  $D = 1 + \beta$ . For the boundary to be fractal, hence

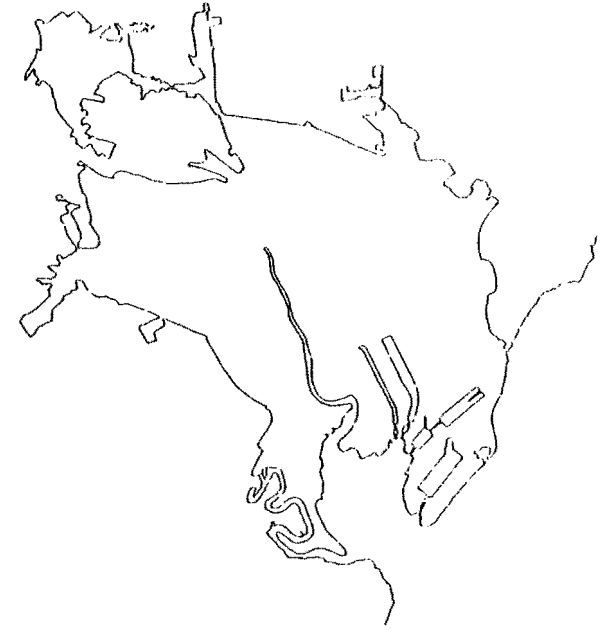
1886

1901

1922

0 1 2 3 4 kms

N



number of boundary points  $n = 2457$   
length of perimeter  $P = 30664$   
average chord length  $P/n = 12.480$

$n = 2757$   
 $P = 30365$   
 $P/n = 11.014$

$n = 4755$   
 $P = 50213$   
 $P/n = 10.560$

Figure 1 Urban boundaries of Cardiff in 1886, 1901 and 1922

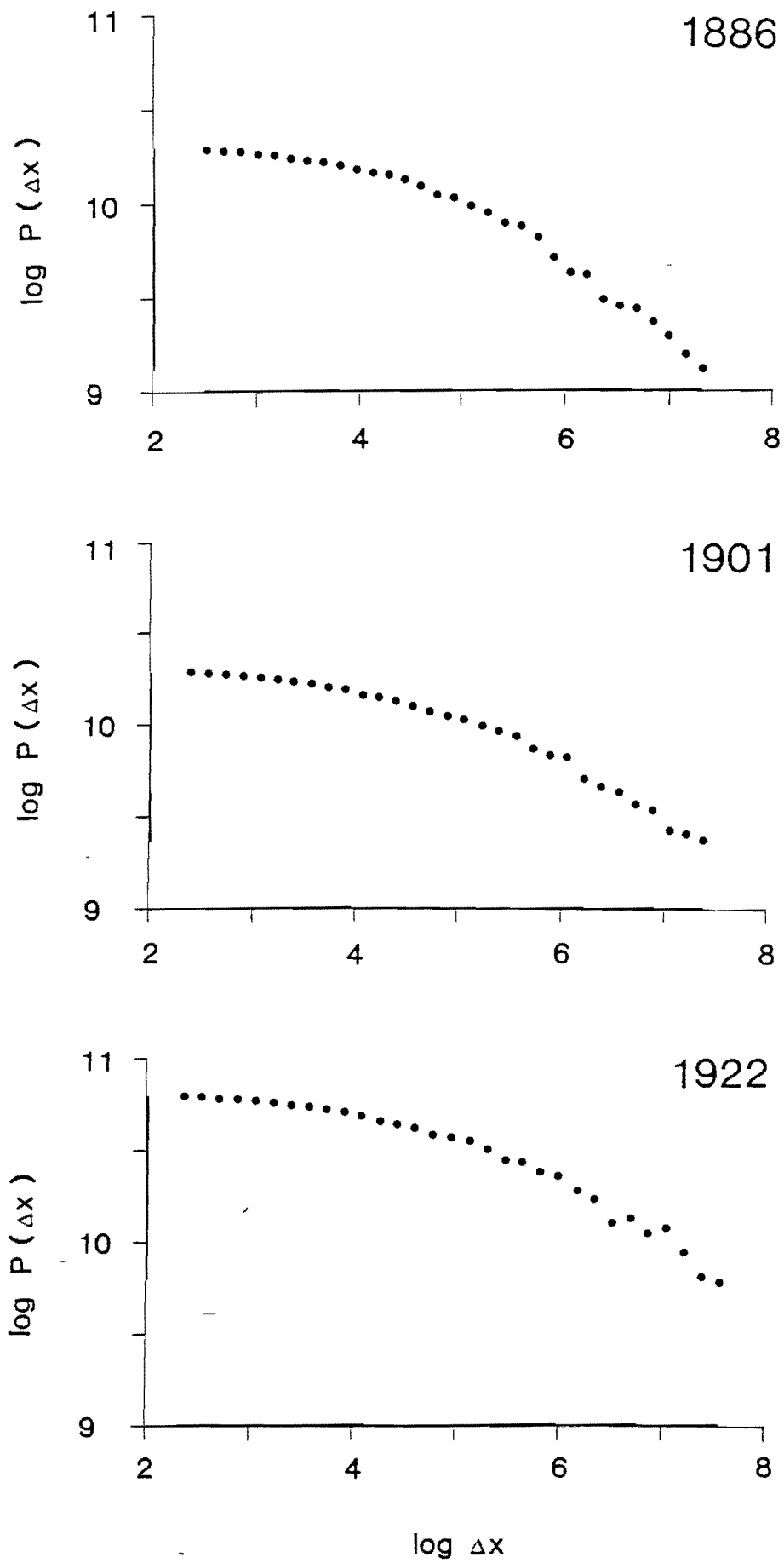


Figure 2 Richardson Plots: variations in perimeter length over many scales

Table 1 Scaling constants and fractal dimensions from equation (2)

Data set	Log $a$	$D = 1 + \beta$	Goodness-of-Fit ( $r^2$ )
1886	11.080	1.239	0.914
1901	10.866	1.184	0.927
1922	11.393	1.185	0.907

irregular,  $1 < D < 2$ , thus the scale coefficient  $\beta$  lies between 0 and 1. A wide variation in the value of  $D$  for geophysical boundaries has been recorded by Burrough (1981) but for coastlines the range is  $1 < D < 1.3$  as first shown by Richardson (1961) and confirmed many times since (Kent and Wong 1982; Mandelbrot 1983; Shelberg, Moellering and Lam 1982).  $a$  and  $\beta$  are determined by a linear regression of  $\log P(\Delta x)$  on  $\log \Delta x$  which from equation (1) is given by

$$\log P(\Delta x) = \log a - \beta \log \Delta x \quad (2)$$

with the sign and size of  $\beta$  determined from the analysis.

Regression based on the values plotted in Figure 2—the so-called ‘Richardson plots’—give the results presented in Table 1. The fractal dimensions  $D$  decrease as hypothesised with the largest fall in the period 1886–1901. However both Figure 2 and Table 1 reveal that the phenomena are multi-fractal. It is impossible to identify clear breaks in the slopes of the plots and thus approximating the plots by several linear functions would be arbitrary. It would appear that the fractal dimension itself is a function of scale and thus we have postulated that the scaling coefficient  $\beta$  is determined as

$$\beta = \lambda + \Phi \Delta x \quad (3)$$

Substituting (3) into (2) gives

$$\log P(\Delta x) = \log a - \lambda \log \Delta x - \Phi \Delta x \log \Delta x \quad (4)$$

from which it is clear that

$$D = 1 + \lambda + \Phi \Delta x \quad (5)$$

As the scale  $\Delta x \rightarrow 0$ ,  $D \rightarrow 1 + \lambda$ . Thus the term  $\Phi \Delta x \log \Delta x$  in equation (4) acts as a dispersion factor which increases the fractal dimension as the scale increases. If  $\Phi = 0$ ,

Table 2 Scaling constants and fractal dimensions from equation (4)

Data Set	Log $a$	$D = 1 + \lambda$ ( $\Delta x = 0$ )	$\Phi \times 10^{-5}$	Goodness-of-Fit ( $r^2$ )
1886	10.719	1.141	5.865	0.983
1901	10.622	1.117	3.947	0.985
1922	11.114	1.109	3.901	0.984

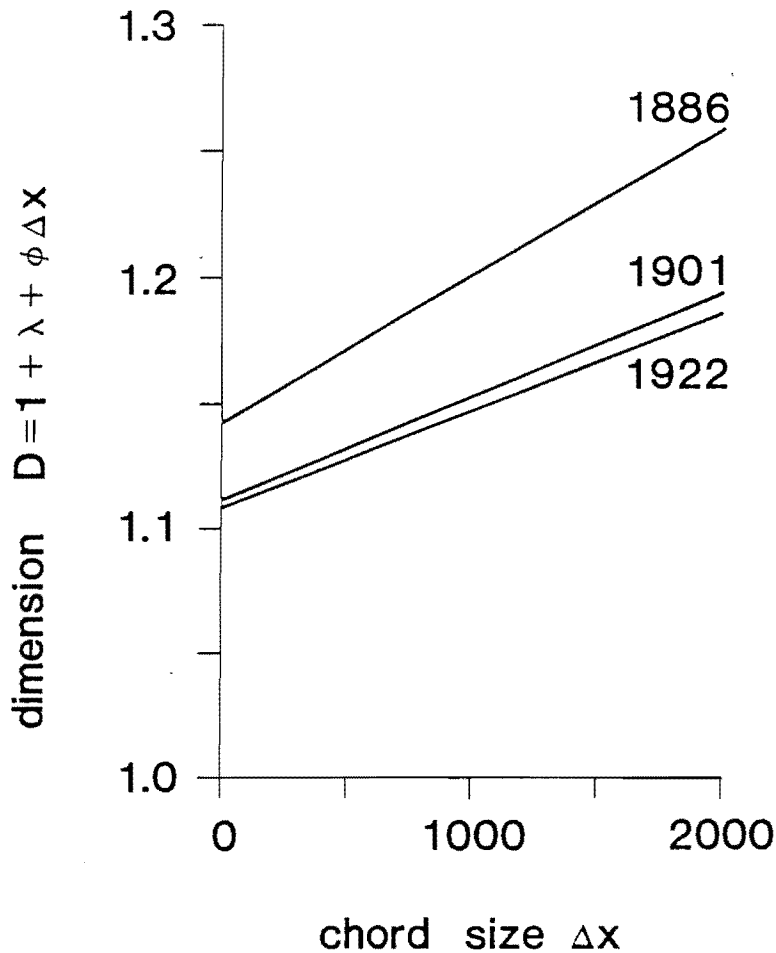


Figure 3 Predicted variations in fractal dimension over scale

then this factor which introduces the non-linearity into the plots is redundant and equation (4) collapses back to equation (2). The model is thus consistent with increasing fractal dimension with scale.

Regressions based on equation (4) are shown in Table 2 and the performance of each model measured by  $r^2$  dramatically improves in comparison with equation (2) and Table 1. Changes in the fractal dimensions based on equation (5) are plotted in Figure 3 from which it is quite clear that the smallest scale dimension where  $\Delta x = 0$ , declines over time in the manner hypothesised. The effect of scale given by  $\Phi$  also decreases over time and in both cases, the greatest decreases in  $\lambda$  and  $\Phi$  occur between 1886 and 1901 when the greatest changes in transport technology—new docks and tramways—were developed.

### Discussion and conclusions

These results are consistent with the three hypotheses originally stated although the decrease in the irregularity of Cardiff's urban boundary between 1886 and 1922 cannot be specifically attributed to changes in any single process of development. However the traditional image of urban growth becoming more irregular as tentacles of development occur around transport lines is not borne out by this analysis. It would appear that greater social and physical controls over development in the late 19th and early 20th century city together with increased accessibility due to improvements in transport,

have combined to gradually reduce the irregularity of urban areas such as Cardiff. These results will apply only to West European cities and similar analyses of North American and other world cities are required. It is tempting to speculate that these results reflect the general notion of increasing controls over the environment but such a conclusion should be avoided because at present there is greater variation in the dimensions produced by different methods than by different temporal data sets on the same city (Batty and Longley 1987).

As a next step in this research, it is necessary to postulate fractal models based on processes which operate at different scales thus generating multi-fractal geometries. Nakano (1983) has indicated how this is possible for a coastline and Suzuki (1984) has demonstrated how such geometries can emerge theoretically over time. These ideas involve the notion of transient self-similarity and transfer the analysis to models of varying self-similarity with respect to morphology and scale. In empirical terms, it may be possible to examine detailed changes in the form of a city, developing an incremental model of urban change in which changes in shape through the boundary are associated with different processes, different degrees of irregularity, different fractal dimensions and at different times. However to generate stronger conclusions will require a longer time series but the most important conclusion here relates to the form of the fractal model. Quite clearly, fractal dimensions of urban boundaries are a function of scale. Other published data such as that pertaining to coastlines and fine particle morphologies should be re-examined in the light of this argument.

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